

h -exponential change of Finsler metric

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Abstract

In this paper, we studied a Finsler space whose metric is given by an h -exponential change and obtain the Cartan connection coefficients for the change. We also find the necessary and sufficient condition for an h -exponential change of Finsler metric to be projective.

Keywords: Finsler space, h -exponential change, projective change.

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1 Introduction

Let $F^n = (M^n, L)$ be an n -dimensional Finsler space equipped with the Fundamental function $L(x, y)$. The metric tensor, angular metric tensor and Cartan tensor are defined by $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$, $h_{ij} = g_{ij} - l_i l_j$ and $C_{ijk} = \frac{1}{2}\dot{\partial}_i g_{jk}$ respectively, where $\dot{\partial}_k = \frac{\partial}{\partial y^k}$. The Cartan connection is given by $CT = (F_{jk}^i, N_k^i, C_{jk}^i)$. The h - and v -covariant derivatives $X_{i|j}$ and $X_i|_j$ of a covariant vector field X_i are defined by [7, 9]

$$(1.1) \quad X_{i|j} = \partial_j X_i - N_j^r \dot{\partial}_r X_i - X_r F_{ij}^r,$$

and

$$(1.2) \quad X_i|_j = \dot{\partial}_j X_i - X_r C_{ij}^r,$$

where $\partial_k = \frac{\partial}{\partial x^k}$.

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In 2012, H. S. Shukla et.al.[10] considered a Finsler space $\overline{F}^n = (M^n, \overline{L})$, whose Fundamental metric function is an exponential change of Finsler metric function given by

$$(1.3) \quad \overline{L} = L e^{\frac{\beta}{L}},$$

where $\beta = b_i(x)y^i$ is 1-form on manifold M^n .

H. Izumi [4] introduced the concept of an h -vector $b_i(x, y)$ which is v-covariant constant with respect to the Cartan connection and satisfies $L C_{ij}^h b_h = \rho h_{ij}$, where ρ is a non-zero scalar function and C_{jk}^i are components of Cartan tensor. Thus if b_i is an h -vector then

$$(1.4) \quad (i) b_i|_k = 0, \quad (ii) L C_{ij}^h b_h = \rho h_{ij}.$$

From the above definition, we have

$$(1.5) \quad L \dot{\partial}_j b_i = \rho h_{ij},$$

which shows that b_i is a function of directional argument also. H. Izumi [4] proved that the scalar ρ is independent of directional argument. Gupta and Pandey [3] proved that if the h -vector b_i is gradient then the scalar ρ is constant. M. Matsumoto [6] discussed the Cartan connection of Randers change of Finsler metric, while B. N. Prasad [8] obtained the Cartan connection of $(M^n, {}^*L)$ where ${}^*L(x, y)$ is given by ${}^*L(x, y) = L(x, y) + b_i(x, y)y^i$, and $b_i(x, y)$ is an h -vector. Gupta and Pandey [1, 2] discussed the hypersurface of a Finsler space whose metric is given by certain transformation with an h -vector.

In the present paper, we consider a Finsler space ${}^*F^n = (M^n, {}^*L)$, whose metric function *L , an h -exponential change of metric, is given by

$$(1.6) \quad {}^*L = L e^{\frac{\beta}{L}},$$

where $\beta = b_i(x, y)y^i$ and b_i is an h -vector. And we obtain the relation between Cartan connection coefficients of F^n and ${}^*F^n$. We also derive the condition for an h -exponential change of metric to be projective.

2 Finsler space ${}^*F^n = (M^n, {}^*L)$

We shall use following notations $L_i = \dot{\partial}_i L = l_i$, $L_{ij} = \dot{\partial}_i \dot{\partial}_j L$, $L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L$. The quantities corresponding to ${}^*F^n$ is denoted by asterisk over that quantity.

From (1.6), we have

$$(2.1) \quad {}^*L_i = e^\tau (m_i + l_i).$$

$$(2.2) \quad {}^*L_{ij} = e^\tau (1 + \rho - \tau) L_{ij} + \frac{e^\tau}{L} m_i m_j.$$

$$(2.3) \quad \begin{aligned} {}^*L_{ijk} = e^\tau (1 + \rho - \tau) L_{ijk} + (\rho - \tau) \frac{e^\tau}{L} [m_i L_{jk} + m_j L_{ik} + m_k L_{ij}] \\ - \frac{e^\tau}{L^2} [m_j m_k l_i + m_i m_k l_j + m_i m_j l_k - m_i m_j m_k], \end{aligned}$$

where $\tau = \frac{\beta}{L}$, $m_i = b_i - \tau l_i$. The normalised supporting element, the metric tensor and Cartan tensor of *F are obtained as

$$(2.4) \quad {}^*l_i = e^\tau (m_i + l_i),$$

$$(2.5) \quad {}^*g_{ij} = \nu e^{2\tau} g_{ij} + e^{2\tau} (2\tau^2 - \tau - \rho) l_i l_j + e^{2\tau} (1 - 2\tau) (b_i l_j + b_j l_i) + 2e^{2\tau} b_i b_j,$$

$$(2.6) \quad {}^*C_{ijk} = \nu e^{2\tau} C_{ijk} + \frac{2}{L} e^{2\tau} m_i m_j m_k + \frac{1}{2L} e^{2\tau} (2\nu - 1) (m_i h_{kj} + m_j h_{ki} + m_k h_{ij}),$$

where $\nu = 1 + \rho - \tau$.

For the computation of the inverse metric tensor, we use the following lemma [5]:

Lemma 2.1. *Let (m_{ij}) be a non-singular matrix and $l_{ij} = m_{ij} + n_i n_j$. The elements l^{ij} of the inverse matrix and determinant of the matrix (l_{ij}) are given by*

$$l^{ij} = m^{ij} - (1 + n_k n^k)^{-1} n^i n^j, \quad \det(l_{ij}) = (1 + n_k n^k) \det(m_{ij})$$

respectively, where m^{ij} are elements of inverse matrix (m_{ij}) and $n^k = m^{ki} n_i$.

The inverse metric tensor of ${}^*F^n$ is derived as follows:

$$(2.7) \quad {}^*g^{ij} = \frac{e^{-2\tau}}{\nu} \left[g^{ij} - \frac{1}{m^2 + \nu} b^i b^j + \frac{\tau - \nu}{m^2 + \nu} (b^i l^j + b^j l^i) - l^i l^j \left\{ \frac{\tau - \nu}{m^2 + \nu} (m^2 + \tau) - \rho \right\} \right],$$

where b is magnitude of the vector $b^i = g^{ij}b_j$.

From (2.6) and (2.7), we obtain

$$\begin{aligned}
(2.8) \quad {}^*C_{ij}^h = & C_{ij}^h + \frac{1}{m^2 + \nu} C_{ijk} b^k (-b^h + 2\tau l^h - \rho l^h - l^h) \\
& + \frac{2}{\nu L} \left[m_i m_j m^h + \frac{1}{m^2 + \nu} m_i m_j m^2 (-b^h + 2\tau l^h - \rho l^h - l^h) \right] \\
& + \frac{1}{2\nu L} (2\nu - 1) \left[m_i h_j^h + m_j h_i^h + m^h h_{ij} \right. \\
& \left. + \frac{1}{m^2 + \nu} (-b^h + 2\tau l^h - \rho l^h - l^h) (2m_i m_j + m^2 h_{ij}) \right].
\end{aligned}$$

3 Cartan connection of the space ${}^*F^n$

Let $C^*\Gamma = ({}^*F_{jk}^i, {}^*N_j^i, {}^*C_{jk}^i)$ be the Cartan connection for the Finsler space ${}^*F^n = (M^n, {}^*L)$.

Since $L_{i|j} = 0$ for the Cartan connection, we have

$$(3.1) \quad \partial_j L_i = L_r F_{ij}^r + \dot{\partial}_r L_i N_j^r.$$

Differentiating (2.1) with respect to x^j , and using (1.1) and (3.1), we get

$$(3.2) \quad {}^*L_{ir} {}^*N_j^r + {}^*L_r {}^*F_{ij}^r = \left[e^\tau \nu L_{ir} + \frac{e^\tau}{L} m_r m_i \right] N_j^r + \left[e^\tau (m_r + l_r) \right] F_{ij}^r + \frac{e^\tau \beta_j m_i}{L} + e^\tau b_{i|j}.$$

Equation (3.2) serves the purpose to find relation between cartan connection of ${}^*F^n$ and F^n . For this, we put

$$(3.3) \quad D_{jk}^i = {}^*F_{jk}^i - F_{jk}^i.$$

With the help of (3.3), the equation (3.2) becomes

$$(3.4) \quad \left[e^\tau \nu L_{ir} + \frac{e^\tau}{L} m_i m_r \right] D_{0j}^r + \left[e^\tau (m_r + l_r) \right] D_{ij}^r = \frac{e^\tau \beta_{|j} m_i}{L} + e^\tau b_{i|j},$$

where the subscript '0' denote the contraction by y^i .

Differentiating (2.2) with respect to x^k , and using (1.1) and (3.1), we have

$$\begin{aligned}
(3.5) \quad & e^\tau \nu \left[L_{ijr} D_{0k}^r + L_{rj} D_{ik}^r + L_{ir} D_{jk}^r \right] + (\nu - 1) \frac{e^\tau}{L} \left[m_r L_{ij} + m_i L_{jr} + m_j L_{ir} \right] D_{0k}^r \\
& - \frac{e^\tau}{L^2} \left[m_i m_j l_r + m_j m_r l_i + m_r m_i l_r - m_i m_j m_r \right] D_{0k}^r + \frac{e^\tau}{L} \left[m_r m_j D_{ik}^r + m_i m_r D_{jk}^r \right] \\
& - \frac{e^\tau (\nu - 1)}{L} L_{ij} \beta_{|k} - \frac{e^\tau}{L^2} \beta_{|k} m_i m_j - e^\tau \rho_k L_{ij} = 0,
\end{aligned}$$

where $\rho_k = \rho_{|k} = \partial_k \rho$.

Theorem 3.1. *The Cartan connection of $*F^n$ is completely determine by the equations (3.4) and (3.5) .*

To prove this, first we propose the following lemma :

Lemma 3.1. *System of equations*

$$\begin{aligned} (i) \quad & *L_{ir} A^r = B_i \\ (ii) \quad & *L_r A^r = B \end{aligned}$$

has unique solution A^r for given B and B_i .

Proof. Using (2.2), equation (i) becomes

$$(3.6) \quad \frac{e^\tau}{L} \left[\nu (g_{ir} - l_i l_r) + m_i m_r \right] A^r = B_i .$$

Contracting by b^i , we get

$$(3.7) \quad m_r A^r = \frac{LB_\beta}{e^\tau} (m^2 + \nu)^{-1} ,$$

here we used subscript β to denote the contraction by b^i , i.e. $B_\beta = B_i b^i$.

From (2.1) and (ii), we have

$$(3.8) \quad l_r A_r = \frac{B}{e^\tau} - \frac{LB_\beta}{e^\tau} (m^2 + \nu)^{-1} .$$

Using (3.7) and (3.8), equation(3.6) becomes

$$g_{ir} A^r = \frac{LB_i}{\nu e^\tau} + l_i \left[\frac{B}{e^\tau} - \frac{LB_\beta}{e^\tau} (m^2 + \nu)^{-1} \right] - \frac{m_i LB_\beta}{\nu e^\tau} (m^2 + \nu)^{-1} ,$$

contracting by g^{ij} , we have

$$(3.9) \quad A^j = \frac{LB^j}{\nu e^\tau} + l^j \left[\frac{B}{e^\tau} - \frac{LB_\beta}{e^\tau} (m^2 + \nu)^{-1} \right] - \frac{m^j LB_\beta}{\nu e^\tau} (m^2 + \nu)^{-1} ,$$

which is concrete form of the solution A^j . □

Now we are in the position to prove the theorem. We will find an explicit expression of difference tensor D_{jk}^i in three steps. Firstly, we will find D_{00}^i and then D_{0k}^i and in the last D_{jk}^i .

Taking symmetric and skew-symmetric part of (3.4), we have

$$(3.10) \quad \begin{aligned} 2e^\tau(m_r + l_r)D_{ij}^r + \left[\nu e^\tau L_{ir} + \frac{e^\tau}{L}m_i m_r\right]D_{0j}^r + \left[\nu e^\tau L_{jr} + \frac{e^\tau}{L}m_j m_r\right]D_{0i}^r \\ = \frac{e^\tau}{L}(\beta_{|j}m_i + \beta_{|i}m_j) + 2e^\tau E_{ij}, \end{aligned}$$

and

$$(3.11) \quad \begin{aligned} \left[\nu e^\tau L_{ir} + \frac{e^\tau}{L}m_i m_r\right]D_{0j}^r - \left[\nu e^\tau L_{jr} + \frac{e^\tau}{L}m_j m_r\right]D_{0i}^r \\ = \frac{e^\tau}{L}(\beta_{|j}m_i - \beta_{|i}m_j) + 2e^\tau F_{ij}, \end{aligned}$$

where $2E_{ij} = b_{j|i} + b_{i|j}$, $2F_{ij} = b_{i|j} - b_{j|i}$.

Contracting (3.10) and (3.11) by y^j , we get

$$(3.12) \quad 2e^\tau(m_r + l_r)D_{0i}^r + \left[\nu e^\tau L_{ir} + \frac{e^\tau}{L}m_i m_r\right]D_{00}^r = \frac{e^\tau}{L}\beta_{|0}m_i + 2e^\tau E_{i0},$$

and

$$(3.13) \quad \left[\nu e^\tau L_{ir} + \frac{e^\tau}{L}m_i m_r\right]D_{00}^r = \frac{e^\tau}{L}\beta_{|0}m_i + 2e^\tau F_{i0},$$

which may be re-written as

$$(3.14) \quad {}^*L_{ir}D_{00}^r = \frac{e^\tau}{L}\beta_{|0}m_i + 2e^\tau F_{i0},$$

where $\beta_{|0} = \beta_{|j}y^j$. Transvecting (3.13) by m^i , we obtain

$$(3.15) \quad m_r D_{00}^r = (m^2 + \nu)^{-1}(\beta_{|0}m^2 + 2LF_{\beta 0}).$$

Contracting (3.12) by y^i , we get

$$2e^\tau(m_r + l_r)D_{00}^r = 2e^\tau E_{00}.$$

i.e.

$$(3.16) \quad {}^*L_r D_{00}^r = e^\tau E_{00}.$$

Applying Lemma 3.1 in equation (3.14) and (3.16), we have

$$(3.17) \quad \begin{aligned} D_{00}^i = \frac{L}{\nu e^\tau} \left[\frac{e^\tau}{L}\beta_{|0}m^i + 2e^\tau F_0^i \right] + l^i \left[E_{00} - \frac{L}{e^\tau}(m^2 + \nu)^{-1} \left(\frac{e^\tau}{L}\beta_{|0}m^2 + 2e^\tau F_{\beta 0} \right) \right] \\ - \frac{m^i L}{\nu e^\tau} (m^2 + \nu)^{-1} \left[\frac{e^\tau}{L}\beta_{|0}m^2 + 2e^\tau F_{\beta 0} \right]. \end{aligned}$$

Here we used $m^i b_i = m_i m^i = m^2$. Also we note that $E_{00} = E_{ij} y^i y^j = b_{i|j} y^i y^j = (b_i y^i)_{|j} y^j = \beta_{|0}$, $F_0^i = g^{ij} F_{j0}$.

Secondly, applying Christoffel process with respect to indices i, j, k in equation (3.5), we have

$$\begin{aligned}
(3.18) \quad & \nu e^\tau \left[L_{ijr} D_{0k}^r + L_{jkr} D_{0i}^r - L_{kir} D_{0j}^r \right] + 2D_{ik}^r \left[\nu e^\tau L_{jr} + \frac{e^\tau}{L} m_r m_j \right] \\
& + \frac{e^\tau}{L} D_{0k}^r \mathfrak{S}_{(rij)} \left[(\nu - 1) m_r L_{ij} - \frac{m_i m_j l_r}{L} \right] + \frac{e^\tau}{L} D_{0i}^r \mathfrak{S}_{(rjk)} \left[(\nu - 1) m_r L_{jk} - \frac{m_j m_k l_r}{L} \right] \\
& - \frac{e^\tau}{L} D_{0j}^r \mathfrak{S}_{(rki)} \left[(\nu - 1) m_r L_{ki} - \frac{m_k m_i l_r}{L} \right] - e^\tau \left[\rho_k L_{ij} + \rho_i L_{jk} - \rho_j L_{ki} \right] \\
& - (\nu - 1) \frac{e^\tau}{L} (\beta_{|k} L_{ij} + \beta_{|i} L_{jk} - \beta_{|j} L_{ki}) - \frac{e^\tau}{L^2} \left[\beta_{|k} m_i m_j + \beta_{|i} m_j m_k - \beta_{|j} m_k m_i \right] \\
& + \frac{e^\tau}{L^2} \left[m_i m_j m_r D_{0k}^r + m_j m_k m_r D_{0i}^r - m_k m_i m_r D_{0j}^r \right] = 0,
\end{aligned}$$

where $\mathfrak{S}_{(ijk)}$ denote cyclic interchange of indices i, j, k and summation. Contracting by y^k , above equation becomes

$$\begin{aligned}
(3.19) \quad & \nu e^\tau \left[L_{ijr} D_{00}^r - L_{jrr} D_{0i}^r + L_{irr} D_{0j}^r \right] + 2D_{0i}^r \left[\nu e^\tau L_{jr} + \frac{e^\tau}{L} m_r m_j \right] \\
& + \frac{e^\tau}{L} D_{00}^r \mathfrak{S}_{(rij)} \left[(\nu - 1) m_r L_{ij} - \frac{m_i m_j l_r}{L} \right] - \frac{e^\tau}{L} D_{0i}^r \frac{m_r m_j l_k}{L} y^k \\
& + \frac{e^\tau}{L} D_{0j}^r \frac{m_r m_i l_k}{L} y^k + \frac{e^\tau}{L^2} m_i m_j m_r D_{00}^r - (\nu - 1) \frac{e^\tau}{L} \beta_{|0} L_{ij} \\
& - \frac{e^\tau}{L^2} \beta_{|0} m_i m_j - e^\tau \rho_0 L_{ij} = 0.
\end{aligned}$$

Adding (3.11) and (3.19), we have

$$(3.20) \quad {}^* L_{ir} D_{0j}^r = G_{ij},$$

where

$$\begin{aligned}
(3.21) \quad 2G_{ij} = & \frac{e^\tau}{L} (\beta_{|j} m_i - \beta_{|i} m_j) - e^\tau \nu L_{ijr} D_{00}^r - \frac{e^\tau}{L} D_{00}^r \mathfrak{S}_{(rij)} \left[(\nu - 1) m_r L_{ij} - \frac{m_i m_j m_r}{L} \right] \\
& + 2e^\tau F_{ij} - \frac{e^\tau}{L^2} m_r m_i m_j D_{00}^r + \frac{(\nu - 1)}{L} e^\tau \beta_{|0} L_{ij} + \frac{e^\tau}{L^2} B_0 m_i m_j + e^\tau \rho_0 L_{ij}.
\end{aligned}$$

Equation (3.12) can be written as

$$(3.22) \quad {}^* L_r D_{0j}^r = G_j,$$

where

$$2G_j = \frac{e^\tau}{L} \beta_{|0} m_j + 2e^\tau E_{j0} + \left[-e^\tau \nu L_{jr} - \frac{e^\tau m_j m_r}{L} \right] D_{00}^r.$$

Using (3.13), above equation may be written as

$$(3.23) \quad G_j = e^\tau (E_{j0} - F_{j0}).$$

Applying Lemma 3.1 in equation (3.20) and (3.22), we obtain

$$(3.24) \quad D_{0j}^i = \frac{LG_j^i}{\nu e^\tau} + \frac{l^i}{e^\tau} \left[G_j - LG_{\beta j} (m^2 + \nu)^{-1} \right] - \frac{m^i LG_{\beta j}}{\nu e^\tau} (m^2 + \nu)^{-1}.$$

Finally, the equation (3.10) may be written as

$$(3.25) \quad {}^*L_r D_{ik}^r = H_{ik},$$

where

$$(3.26) \quad \begin{aligned} 2H_{ik} = & \frac{e^\tau}{L} (\beta_{|k} m_i + \beta_{|i} m_k) + e^\tau E_{ik} - \left[e^\tau \nu L_{ir} + \frac{e^\tau}{L} m_i m_r \right] D_{0k}^r \\ & - \left[e^\tau \nu L_{kr} + \frac{e^\tau}{L} m_k m_r \right] D_{0i}^r. \end{aligned}$$

Equation (3.18) may be written as

$$(3.27) \quad {}^*L_{rj} D_{ik}^r = H_{jik},$$

where

$$(3.28) \quad \begin{aligned} 2H_{jik} = & -\nu e^\tau \left[L_{ijr} D_{0k}^r + L_{jkr} D_{0i}^r - L_{kir} D_{0j}^r \right] + e^\tau \left[\rho_k L_{ij} + \rho_i L_{jk} - \rho_j L_{ki} \right] \\ & - \frac{e^\tau}{L} D_{0k}^r \mathfrak{S}_{(rij)} \left[(\nu - 1) m_r L_{ij} - \frac{m_i m_j l_r}{L} \right] - \frac{e^\tau}{L} D_{0i}^r \mathfrak{S}_{(rjk)} \left[(\nu - 1) m_r L_{jk} - \frac{m_j m_k l_r}{L} \right] \\ & + \frac{e^\tau}{L} D_{0j}^r \mathfrak{S}_{(rki)} \left[(\nu - 1) m_r L_{ki} - \frac{m_k m_i l_r}{L} \right] \\ & + (\nu - 1) \frac{e^\tau}{L} (\beta_{|k} L_{ij} + \beta_{|i} L_{jk} - \beta_{|j} L_{ki}) + \frac{e^\tau}{L^2} \left[\beta_{|k} m_i m_j + \beta_{|i} m_j m_k - \beta_{|j} m_k m_i \right] \\ & - \frac{e^\tau}{L^2} \left[m_i m_j m_r D_{0k}^r + m_j m_k m_r D_{0i}^r - m_k m_i m_r D_{0j}^r \right]. \end{aligned}$$

Applying Lemma 3.1 in (3.25) and (3.27), we have

$$(3.29) \quad D_{ik}^j = \frac{LH_{ik}^j}{\nu e^\tau} + \frac{l^j}{e^\tau} \left[H_{ik} - LH_{\beta ik} (m^2 + \nu)^{-1} \right] - \frac{m^j L}{\nu e^\tau} H_{\beta ik} (m^2 + \nu)^{-1},$$

where we put $H_{ik}^j = g^{jm} H_{mik}$.

Thus in view of (3.3), we get the Cartan connection coefficient ${}^*F_{jk}^i$. This completes the proof of theorem (3.1).

Now, suppose Cartan connection coefficients for both spaces F^n and $*F^n$ are same, i.e. $*F_{jk}^i = F_{jk}^i$. Then $D_{jk}^i = 0$. But then equations (3.12) and (3.13) implies that $E_{i0} = F_{i0}$, and hence

$$(3.30) \quad b_{0|i} = 0,$$

i.e. $\beta_{|i} = 0$. Differentiating $\beta_{|i} = 0$ partially with respect to y^j and applying commutation formulae $\dot{\partial}_j(\beta_{|i}) - (\dot{\partial}_j\beta)_{|i} = -(\dot{\partial}_r\beta)C_{ij|0}^r$, we get

$$(3.31) \quad b_{j|i} = b_r C_{ij|0}^r.$$

From the above equation, we conclude that $F_{ij} = 0$. M. K. Gupta and P. N. Pandey [3] has proved that if h -vector b_i is gradient, i.e. $F_{ij} = 0$ then ρ is constant, i.e. $\rho_i = \rho_{|i} = 0$. Taking h -covariant derivative of $LC_{ij}^r b_r = \rho h_{ij}$ and using $L_{|k} = 0$, $\rho_{|k} = 0$ and $h_{ij|k} = 0$, we have

$$(b_r C_{ij}^r)_k = \frac{\rho}{L} h_{ij} = 0,$$

i.e.,

$$b_{r|k} C_{ij}^r + b_r C_{ij|k}^r = 0.$$

From (3.31), $b_{r|k} = b_{k|r}$ and hence above equation becomes

$$b_{k|r} C_{ij}^r + b_r C_{ij|k}^r = 0.$$

Transvecting by y^k , we have $b_{0|r} C_{ij}^r + b_r C_{ij|0}^r = 0$. Using (3.30) and (3.31), we conclude that $b_{i|j} = 0$.

Conversely, $b_{i|j} = 0$ implies that $E_{ij} = 0 = F_{ij}$ and $\beta_{|i} = \beta_i = b_{j|i} = 0$. $F_{ij} = 0$ implies that $\rho_i = \rho_{|i} = 0$ [3]. Therefore from (3.17), we get $D_{00}^i = 0$ and then $G_{ij} = 0$ and $G_j = 0$. This gives $D_{0j}^i = 0$ and then $H_{jik} = 0$ and $H_{ik} = 0$. Therefore (3.29) implies that $D_{jk}^i = 0$. Thus, we have:

Theorem 3.2. *For an h -exponential change of metric, the Cartan connection coefficients for both spaces F^n and $*F^n$ are same if and only if the h -vector b_i is parallel with respect to Cartan connection of F^n .*

Now transvecting (3.3) by y^j and using $F_{jk}^i y^j = G_k^i$, we obtain

$$(3.32) \quad {}^*G_k^i = G_k^i + D_{0k}^i.$$

Transvecting again the above equation by y^k and using $G_k^i y^k = 2G^i$, we get

$$(3.33) \quad 2 {}^*G^i = 2G^i + D_{00}^i.$$

Differentiating (3.32) partially with respect to y^h and using $\dot{\partial}_h G_k^i = G_{kh}^i$, we have

$$(3.34) \quad {}^*G_{kh}^i = G_{kh}^i + \dot{\partial}_h D_{0k}^i,$$

where G_{kh}^i are Berwald connection coefficients.

Now, if the h -vector b_i is parallel with respect to Cartan connection of F^n then by Theorem (3.2), the Cartan connection coefficients for both spaces F^n and ${}^*F^n$ are same, therefore $D_{jk}^i = 0$. Hence from (3.34), we get ${}^*G_{kh}^i = G_{kh}^i$.

Thus, we have :

Theorem 3.3. *For an h -exponential change of metric, if an h -vector b_i is parallel with respect to Cartan connection of F^n then Barwald connection coefficients for both spaces F^n and ${}^*F^n$ are the same.*

4 Condition for h -exponential change of metric to be projective

Let us consider Finsler spaces $F^n = (M^n, L)$ and ${}^*F^n = (M^n, {}^*L)$. A transformation from L to *L is called projective change if any geodesis on $F^n = (M^n, L)$ is also geodesis on ${}^*F^n = (M^n, {}^*L)$ and vice-versa. A geodesis on F^n is given by

$$\frac{dy^i}{dx} + 2G^i(x, y) = \tau y^i; \quad \tau = \frac{d^2s/dt^2}{ds/dt}$$

The change $L \mapsto {}^*L$ is projective change if and only if there exists a scalar function $P(x, y)$ which is positive homogeneous of degree one in y^i and satisfies

$${}^*G(x, y) = G(x, y) + P(x, y)y^i.$$

Now, we find the condition for exponential change with h -vector to be projective. From (3.33), it follows that exponential change with h -vector to be projective if and only if $D_{00}^i = 2Py^i$. Then from (3.17), we get

$$(4.1) \quad 2Py^i = \frac{L}{\nu e^\tau} \left[\frac{e^\tau}{L} \beta_{|0} m^i + 2e^\tau F_0^i \right] + l^i \left[E_{00} - \frac{L}{e^\tau} (m^2 + \nu)^{-1} \left(\frac{e^\tau}{L} \beta_{|0} m^2 + 2e^\tau F_{\beta 0} \right) \right] - \frac{m^i L}{\nu e^\tau} (m^2 + \nu)^{-1} \left[\frac{e^\tau}{L} \beta_{|0} m^2 + 2e^\tau F_{\beta 0} \right].$$

Transvecting (4.1) by y_i and using $m^i y_i = 0$, $F_0^i y_i = 0$, we get

$$(4.2) \quad P = \frac{y_i l^i}{2L^2} \left[E_{00} - \frac{L}{e^\tau} (m^2 + \nu)^{-1} \left(\frac{e^\tau}{L} \beta_{|0} m^2 + 2e^\tau F_{\beta 0} \right) \right].$$

Substituting the value of P in (4.1), we get

$$(4.3) \quad F_0^i = \frac{m^i}{2L} (m^2 + \nu)^{-1} (\beta_{|0} m^2 + 2L F_{\beta 0}) - \frac{\beta_{|0} m^i}{2L}.$$

Using (3.15) in above equation, we have

$$(4.4) \quad F_0^i = \frac{m^i}{2L} m_r D_{00}^r - \frac{\beta_{|0} m^i}{2L}.$$

Transvecting by g_{ij} to above equation, we have

$$(4.5) \quad F_{i0} = \frac{m_i}{2L} m_r D_{00}^r - \frac{\beta_{|0} m_i}{2L}.$$

Using (4.5) in (3.13) and referring $\nu \neq 0$, we obtain $L_{ir} D_{00}^r = 0$, which transvecting by m^i and using $L_{ir} m^i = \frac{1}{L} m_r$, we get $m_r D_{00}^r = 0$, and then (4.5) becomes

$$(4.6) \quad F_{i0} = -\frac{\beta_{|0} m_i}{2L}.$$

The equation (4.6) is necessary condition for h -exponential change to be projective change.

Conversely, if (4.6) satisfied, the equation (3.13) yields

$$(4.7) \quad \left[e^\tau \nu L_{ir} + \frac{e^\tau}{L} m_i m_r \right] D_{00}^r = 0.$$

Transvecting by m^i and referring $(m^2 + \nu) \neq 0$, we get $m_r D_{00}^r = 0$ and then (3.17) gives $D_{00}^i = E_{00} l^i$. Therefore $*F^n$ is projective to F^n . Thus, we have:

Theorem 4.1. *The h -exponential change given by (1.6) is projective if and only if condition (4.6) is satisfied.*

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